cants for relief, to assist them in distress, and to procure them employment.

What a kind husband, what a loving father, Mr. Gompertz was, only the immediate members of his family know: those to whom he extended his friendship were at a loss which more to admire, his social qualities or mental attainments.

May science ever find votaries as worthy and ardent as the late Benjamin Gompertz!

On the Limitation of Risks; being an Essay towards the Determination of the Maximum Amount of Risk to be retained by a Life Insurance Company on a Single Contingency. By THOMAS BOND SPRAGUE, M.A., *Vice-President of the Institute of Actuaries.*

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IT is my object in the present paper to discuss the principles which should guide the actuary in fixing the maximum amount of risk to be retained by a Life Insurance Company upon a single contingency.

Those principles are by no means limited to life insurance, but extend with suitable modifications to other kinds of insurance, as fire, marine, hailstorm, health, and accidental death insurance. They are also applicable to the risks of mercantile transactions, and in particular to the risks incurred by a bank in discounting its customers' bills. There can be little doubt that some recent failures of banks have been caused principally by the neglect of these principles; in other words, the manager has failed to put a proper limit on the risks undertaken, *i. e.*, has trusted individuals to an imprudent extent, and the loss of heavy sums consequent on their failure has proved disastrous to the bank.

The risks attending life insurance however differ in several important respects from the other classes of risks above mentioned. In particular, there cannot be a "partial loss," as there may be in those other businesses. An ordinary life policy differs also from other kinds of insurance in the following respects—that the contract entered into by the Office issuing the policy, does not run out in a fixed time, and cannot be terminated at the option of the Office at stated periods like a fire insurance contract. Again,

the risk increases yearly during the continuance of the policy; and lastly, it is certain that in every case a claim will sooner or later arise. For these reasons it will be convenient to limit our attention on this occasion to the risks of life insurance only.

As a first step in the discussion, let us seek an answer to the inquiry:—"Why is it necessary for a Life Office to put any limit at all upon the risks it undertakes? Why should not an Office, for example, issue a policy for £100,000, and run the risk for the whole amount?" At first sight it may seem sufficient to answer, that no Office should run the risk of being called on to pay a sum which it would be impossible or inconvenient for it to provide. But very little consideration is required to show that this is not a satisfactory or a complete reply. For every Insurance Company does, and must, from the nature of its business, run the risk of being called on to pay sums which it would be impossible for it to realize. If all the policies issued by any Company, or even a large proportion of them, were to become claims simultaneously, the Company would assuredly be ruined. But this chance of ruinous loss is one which the Company may properly incur. To take an example, it cannot be denied that a Company with a paid-up capital of £20,000 would be quite justified in becoming liable for £100,000 under 100 policies of £1,000 each, or under 1,000 policies of £100 each; but it would not be justified in becoming liable for the same sum under 10 policies of £10,000 each, and certainly not under 2 policies of £50,000, or a single one of £100,000.

What is it, then, that constitutes the difference between these various cases? We will suppose for the present that all the insurances are for a single year only. Then the most obvious difference is that the more numerous the insurances are which constitute the total risk, £100,000, the less is the chance of the Company being called on to pay the whole. If the rate of mortality be taken at 1 per cent. per annum,* or the chance of dying

* The mortality being such that out of 1,000 persons alive at the beginning of a year 10 die during the year, it appears correct to say, either that the "annual mortality is one per cent.," or that "the rate of mortality is one per cent. per annum," or that "the annual rate of mortality is ·01." In the latter forms of expression, the phrase "rate of mortality" is used in a sense strictly analogous to the common one "rate of interest." Thus we say that "the rate of interest is five per cent. per annum," if £1,000 has by the operation of interest become £1,050 at the end of a year.

It must be borne in mind by the student, in reading the Reports of the Registrar-General and similar works, that the term "rate of mortality" is used by Dr. Farr—most injudiciously and improperly in my opinion—to denote a different quantity:—that, namely, which has been called by Mr. Gompertz the "intensity of mortality," and by Mr. Woolhouse (very happily) the "force of mortality." It is not easy to give in

in a year = ·01, then the probability of the Company being called on to pay £100,000 under a single policy for that amount is ·01. If there are 10 policies of £10,000 each, on different lives, the probability of their all falling in will be $\cdot 01^{10}$; and if there are 100 policies of £1,000 each, the probability of the Office being called on for the whole £100,000 is $\cdot 01^{100}$, which is an excessively small quantity with 199 ciphers following the decimal point.

This, however, is a very incomplete comparison; and we shall get a better idea of the difference between the various cases, if we calculate in each the chance that the loss will be not less than a given amount—say £10,000. Thus, when there are 10 policies of £10,000 each, the chance that there will be no claim is $\cdot 99^{10}$, which is equal to ·90438; and therefore the probability of at least one loss occurring is ·09562, which is the probability that the Office will be called on to pay £10,000, or more.

Next, when there are 100 policies of £1,000 each, denoting by P_x the probability that x claims will arise, we have

popular language a strict definition of the "force of mortality;" but it is approximately equal to the ratio which the number dying in a year bears to the number living in the middle of the year.

To enable the reader to understand the distinction more readily, I quote the remarks of Dr. Farr, given in the Blue Book, on the Sanitary State of the Army in India (p. 37):—"In the reports upon the Indian Fund, the probability of dying is incorrectly called the *rate of mortality*, so as to mislead the unwary reader. Thus, if on an average out of 100 men living at the beginning of a year there are 10 deaths in the year following, the probability of dying is expressed by the fraction $\frac{10}{100}$, which is incorrectly called in the reports 'the rate of mortality'; but the rate of mortality is $\frac{10}{95}$, for the numbers living at the end of the year are 90, and the years of life are $90+10$ half years, which it may be assumed are lived by the 10 who died in the course of the year." This appears to me much as if we should say, "If £100 has by the operation of interest become £105 at the end of a year, the rate of interest is not $\frac{5}{100}$, but $\frac{5}{102\frac{1}{2}}$." In my opinion Dr. Farr is wrong, and the authors he criticizes are right, in the meaning they attach to the phrase, "rate of mortality."

I have not been able to ascertain that any writer, with the exception of Mr. Edmonds, agrees with Dr. Farr in his views on this point; and I conclude that the attempt to attach a new meaning to the familiar phrase, "rate of mortality," is an innovation much to be deplored, and in every way to be discountenanced. By this and similar innovations—as his definition of an "Annuity," and his alteration of the "N Column"—Dr. Farr has greatly diminished the usefulness of his writings.

If L_x denote the number living at any age x, integral or fractional, then the annual "rate of mortality" at that age x will be $\frac{L_x - L_{x+1}}{L_x}$; and the "force of mortality" at the same age, $\frac{1}{L_x} \cdot \frac{dL_x}{d_x}$. If the mortality follow Gompertz's law so that $L_x = dg^{q^x}$, then the annual rate of mortality at the age x will be found to be $1 - g^{(q-1)q^x}$; and the force of mortality, $\log_e g . \log_e q . q^x$.

P_0* = ·3660323	P_5 = ·0028978
P_1 = ·3697296	P_6 = ·0004634
P_2 = ·1848648	P_7 = ·0000628
P_3 = ·0609992	P_8 = ·0000074
P_4 = ·0149417	P_9 = ·0000007
	·9999997

The sum of these probabilities being

Here we see that the chance that there will be no claim is rather more than one in three; the chance of a single claim arising is also rather more than one in three; and the chance of more than one claim arising is less than one in three. We see also that the probability of the losses amounting to £10,000 or more, is about ·0000003.

This example shows us how rapidly the probability of a dangerous loss diminishes when the number of risks is increased. On the other hand, it must be noticed that the probability that no loss will arise, diminishes rapidly as the number of risks increases. In the case of a single policy, the chance of there being no loss is ·99. When there are ten policies, the probability that there will be no claim is $\cdot 99^{10}$ or ·90438. When there are 100 and 1,000 policies, the same probabilities are ·3660323 and ·000043171 respectively.

Thus then we see that, as the number of policies is increased—the sum assured remaining the same—the probability of a large loss occurring is diminished, and also the probability of a very small loss occurring. The effect of increasing the number of

* These values have been calculated as follows:—$P_0 = \cdot 99^{100}$, and is easily found by logarithms. Then

$$P_1 = \frac{100}{1} \times \cdot 99^{99} \times \cdot 01 = \frac{100}{99} P_0 = \frac{100}{100-1} P_0$$

$$= \frac{P_0}{1 - \frac{1}{100}} = P_0 \left\{ 1 + \frac{1}{100} + \frac{1}{(100)^2} + \ldots \ldots \right\}$$

$$= P_0 + \cdot 01 P_0 + \cdot 0001 P_0 + \&c.$$

$$P_2 = \frac{100 \times 99}{1 \times 2} \times \cdot 99^{98} \times \cdot 01^2 = \frac{1}{2} \times \cdot 99^{99} = \frac{P_1}{2}$$

$$P_3 = \frac{100 \times 99 \times 98}{1 \times 2 \times 3} \times \cdot 99^{97} \times \cdot 01^3 = \frac{98}{3} \times \frac{1}{99} P_2$$

$$= \frac{1}{3} \times \frac{99-1}{99} \times P_2 = \frac{1}{3} \left(1 - \frac{1}{99} \right) P_2$$

$$= \frac{1}{3} \{ 1 - \cdot 01 - \cdot 0001 - \&c. \ldots \ldots \} P_2$$

$$P_4 = \frac{97}{4} \times \frac{1}{99} \times P_3$$

&c. &c.

policies, is to diminish the probability of a large departure from the average in either direction; and the larger the number of lives under observation, the less is the chance of an accidental fluctuation in the number of deaths.

It is not my intention on the present occasion to pursue our subject further in this direction, and to seek by the aid of the higher mathematics, a measure of the rate at which the probability of a fluctuation decreases when the number of cases is increased, as is done by De Morgan in his treatises on the *Theory of Probabilities*. I have to confess, indeed, that I do not consider the results of those investigations of much practical importance in their application to the theory of life contingencies. In fact, we must always be careful in reasoning on that subject that we do not apply our mathematical conclusions too unreservedly; we must always bear carefully in mind the suppositions and limitations under which they have been obtained. For example, if we trace from year to year the number of deaths in the population at large, we shall often find greater fluctuations than the theory of probabilities would lead us to expect—the fact being that our fundamental supposition of the probability of death being the same in different years at the same age, is rendered untrue by the occurrence of unhealthy years, or years of scarcity and distress. And as the number of persons under observation in a Life Insurance Company increases and becomes very large, we may expect that while accidental fluctuations arising from the paucity of members will become of less consequence; on the other hand, fluctuations may be anticipated from the same causes as produce them in the number of deaths among the population at large.

The practical conclusion at which I have arrived then is that in all Insurance Companies, whether small or large, fluctuations in the mortality must be expected; and a proper provision should be made for them in the premiums charged.

We have seen that the effect of increasing the number of policies, the total sum assured remaining the same, is to lessen the probability of a very large loss or of a very small loss occurring; and we shall find that the "expectation of loss" remains unchanged, being in all the cases we have considered above, equal to the premium for the risk, £1,000. It will be worth while to elucidate this point further.

1. When there is a single policy for £100,000 the Office can only lose the whole or nothing. The chance of losing the whole is ·01; and the expectation of loss = ·01 × £100,000 = £1,000.

2. When there are 10 policies for £10,000 each, the Office may lose either

£100,000, £90,000, £80,000, ... £10,000, or nothing;

and the probabilities of these several losses are—

$$\cdot 01^{10},\ 10\times\cdot 01^{9}\times\cdot 99,\ 45\times\cdot 01^{8}\times\cdot 99^{2},\ \ldots\ \cdot 99^{10},$$

being the terms of the expansion of $(\cdot 01+\cdot 99)^{10}$.

3. When there are 100 policies of £1,000 each, the Office may lose either

£100,000, £99,000, £98,000, ... £2,000, £1,000, or nothing;

and the probabilities of these several losses are the terms of the expansion of $(\cdot 01+\cdot 99)^{100}$.

We have now to prove that in these latter cases, as also when there are 1,000 policies of £100 each, the expectation of loss is still £1,000. In order to do this, let us suppose that a sum S is assured under any number n of policies; and that the chance of any one becoming a claim is x, then the possible losses are—

$$\mathrm{S},\ \frac{n-1}{n}\cdot\mathrm{S},\ \frac{n-2}{n}\cdot\mathrm{S}\ \ldots\ \frac{2\mathrm{S}}{n},\ \frac{\mathrm{S}}{n},\ 0;$$

and the probabilities of these several losses are

$$x^n,\ nx^{n-1}(1-x),\ \frac{n.n-1}{1.2}x^{n-2}(1-x)^2,\ \ldots\ nx(1-x)^{n-1},\ (1-x)^n.$$

The total expectation of loss is therefore

$$\begin{aligned}&\mathrm{S}\times x^n+\frac{n-1}{n}\mathrm{S}\times nx^{n-1}(1-x)+\frac{n-2}{n}\mathrm{S}\times\frac{n.(n-1)}{1\cdot 2}x^{n-2}(1-x)^2\\&\qquad\qquad+\ \ldots\ \ldots\ +\frac{\mathrm{S}}{n}.nx(1-x)^{n-1}\\&=\mathrm{S}x\{x^{n-1}+(n-1)x^{n-2}(1-x)+\frac{(n-1)(n-2)}{1\cdot 2}x^{n-3}(1-x)^2\\&\qquad\qquad+\ \ldots\ \ldots\ +(1-x)^{n-1}\}\\&=\mathrm{S}x\{x+(1-x)\}^{n-1}=\mathrm{S}x\times 1^{n-1}=\mathrm{S}x.\end{aligned}$$

Thus we see that the total expectation of loss is independent of n, the number of policies; and in the case we have been considering, where S = 100,000 and $x=\cdot 01$, the expectation of loss is £1,000, whatever the number of policies.

Before proceeding further we pause to point out that this "expectation of loss" is a different thing from the "most probable loss." In fact, in many cases it is impossible for the Office to lose the exact amount of the "expectation of loss." To take an

example, if £100,000 is insured under 625 policies of £160 each, it is not possible for the Office to lose exactly £1,000. In order to find the amount of the most probable loss, we will resume our former supposition of a sum s, insured under n policies of equal amounts, then the probability of a loss of $\frac{r\mathrm{S}}{n}$ occurring is

$$\frac{n.\overline{n-1}.\overline{n-2}\ \ldots\ \overline{r+1}}{1.2\ \ldots\ \overline{n-r}}\,x^r(1-x)^{n-r},$$

and the probability of a loss of $\frac{r+1}{n}\mathrm{S}$ is

$$\frac{n.\overline{n-1}.\overline{n-2}\ \ldots\ \overline{r+2}}{1.2\ \ldots\ \overline{n-r-1}}\cdot x^{r+1}(1-x)^{n-r-1};$$

and the latter probability is greater or less than the former

according as $\quad \frac{n-r}{r+1}\cdot\frac{x}{1-x} \gtrless 1,$

according as $\quad (n-r)x \gtrless (r+1)(1-x),$

according as $\quad nx-rx \gtrless r+1-rx-x,$

according as $\quad nx+x-1 \gtrless r,$

according as $\quad r \lessgtr nx+x-1.$

Now, as r increases from 0 to n, the fraction $\frac{n-r}{r+1}\cdot\frac{x}{1-x}$, decreases from $\frac{nx}{1-x}$ to 0; and if $\frac{nx}{1-x}>1$, then the fraction will be first >1 and afterwards <1. Hence, as r increases, if $\frac{nx}{1-x}>1$, the probability of r claims arising will first increase and then diminish, and will be greatest when r is the integer next greater than $(n+1)x-1$; or is the integral part of $(n+1)x$. If $(n+1)x$ happen to be an integer, then we conclude that $(n+1)x-1$ and $(n+1)x$ claims are equally probable, and more probable than any other number of claims.

Returning now to the supposition of 625 policies for £160 each, we have $x=625$, $x=\cdot01$, or $(n+1)x=6\cdot26$, so that the most probable number of claims is 6, giving a loss of £960. Here, as was to be expected, the "most probable loss" does not differ much from the "expectation of loss," which we have already seen to be £1,000. In this instance $\frac{nx}{1-x}=625\times\frac{\cdot01}{\cdot99}=\frac{625}{99}$ and is therefore greater than unity.

As an illustration of this case where $(n+1)\,x$ is an integer, let us take $n+1=700$, or $n=699$, so that we have 699 policies of £143·0615 each. Then, since $(n+1)x=7$, we learn from what precedes that 6 and 7 claims are equally probable, and more probable than any other number. In those cases, the losses would be £858·37 and £1,001·43 respectively.

We conclude also, comparing the results of the two last paragraphs, that if the number of policies is between 625 and 699, then the most probable number of claims is 6; and it is easy to see that 6 is the most probable number of claims if the number of the policies be anything from 600 to 698 inclusive.

Again, when there are 1,000 policies of £100 each, we have $(n+1)\,x=10{\cdot}01$, or the most probable number of claims is 10, and the most probable loss £1,000, the same as the expectation of loss. In general, the expectation of loss is $\mathrm{S}x$ or $nx \times \frac{\mathrm{S}}{n}$, and the most probable loss is $\left\{\text{the integral part of } (n+1)\,x\right\} \times \frac{\mathrm{S}}{n}$. If, then, nx is an integer, it is of necessity the integral part of $(n+1)\,x$; and the expectation of loss is equal to the most probable loss.

It will be noticed that the "expectation of loss" is analagous to the "expectation of life at a given age," called by some writers the "mean duration of life." The "most probable loss," is analagous to the "most probable duration of life"—that term of years which is more likely than any other to be the duration of life. This is not the same as the term of years (called by French writers "*la vie probable*") which it is as likely as not that a given person will survive. (See Gray's *Tables and Formulæ*, §§ 162, 163, where the difference between these various quantities is very clearly explained.) To take an example: By the Carlisle Table a person of the age 30, has an "expectation of life" equal to 34·34 years. But since, according to the Carlisle Table, more persons over the age of 4 die at the age of 74 than at any other age, the "most probable duration of life" is 44·5 years. Again; since of the persons living at the age 30, one half attain the age of 66 but not 67, it follows that the time which a person 30 years of age has an even chance of surviving, is a little over 36 years. We have not considered in what precedes the quantity which is analogous to this one; it is "the loss which it is an even chance that the claims of the year will not fall short of."

We return now to the question, "Why is an Office with a capital of £20,000, justified in becoming liable for £100,000

under 100 policies of £1,000 each, but not under 10 policies of £10,000 each?" The preceding investigations enable us to say that the chance of the Office sustaining a dangerous or ruinous loss, is very much less in the former case than in the latter; and is, in fact, so small in the former case, that it may be disregarded.

Can we then conclude generally that a Company is justified in running the risk of having to pay a very large sum on the happening of any extremely improbable contingency? For example, would a Company be justified in granting a policy for £100,000 to be payable only in case a young man, now 20 years of age, should die on his 90th birthday, or in case all of 20 specified persons should die within a year in a specified order? . . . . (A).

It will no doubt be answered by some persons that a Company would not be justified in undertaking insurances of the nature of those mentioned in the last paragraph, because such insurances, if ever made, would be rare, and could never be sufficiently numerous to enable the Company to obtain an average result from them. To this argument I cannot myself allow any weight. I can find no sufficient reason for thinking that it is necessary for an Insurance Company to have a sufficient number of each class of risks, to form a separate average. On this point I entirely agree with the remarks of the late Dr. Young, given in the *Assurance Magazine*, vol. vi., p. 289. "There is a common prejudice that it is disadvantageous for an Office to take a single risk of any particular description; and it is sometimes said that if the adventure should happen to be unsuccessful, there would be no possible compensation from others of the *same* kind. There is however just the same chance that it would be compensated by others of a *different* kind."

I do not propose to discuss this point at the present time, but will content myself with an illustration. In the course of conversation with a merchant, I learned that he did not insure his house or furniture against fire; and on my inquiring the reason, he said, "If my house were burnt I might lose £1,500 or £2,000; but I run as great or greater risks in the ordinary course of my business. I may any day lose £2,000 by the failure of a single customer; and if I run that risk, I may fairly run the risk of loss by fire too." Subsequent consideration of this argument has led me to question its soundness, on the ground of the insufficiency of the consideration for the risk. While admitting that risks of wholly different kinds may fairly be set off the one against the other, I would yet ask, "Is it worth while for a merchant to run the risk

of losing, say £2,000, by fire, his consideration for the risk being the premium of insurance which he saves?" The amount he saved by running his own risk may be 4*s.* 6*d.* or 5*s.* per £100 per annum; but the consideration for the risk of a loss in the ordinary course of trade must have been far larger than this. The ordinary profits of trade cannot, I should suppose, be taken at less than 10 per cent. per annum—subject to deduction for bad debts. I have no means of estimating the probability of bad debts arising, but should suppose it cannot ordinarily exceed 2 or 3 per cent. per annum. Here then there would be a profit of 8 or 7 per cent. per annum, which may be considered as the consideration for incurring the risk of a bad debt; and by the side of this, the consideration of ¼ per cent. for running the risk of loss by fire, seems quite contemptible. On *this* ground, then, but not on account of the diverse characters of the risks, I conclude that it is not worth while for a merchant to be his own insurer against fire.

The conclusion I arrive at, then, on this point, is that risks of various kinds may be fairly classed together, when the amounts of the several risks are approximately equal, and also the several chances of loss. But when there is a great disparity in the chances of loss and in the premiums paid, then the risks cannot properly be classed together. For this reason, therefore, an Insurance Company cannot properly class together with life insurance risks those of fire insurance or accidental death—the premium for the first scarcely ever being less than £1 per cent. per annum; and in the second, commonly 1*s.* 6*d.* per cent. per annum; while in the last, a common premium is 3*d.* per £1,000 or ·00125 per cent. If then a Company transacted business of these three kinds, there should be enough risks of each class to form a separate average.

But on the other hand there seems to be no sufficient reason for the directors of a Life Office to decline to grant an insurance on the life of a negro, mulatto, or native Indian, on the ground that it is improbable there will be many such insurances effected.

We are now in a position to see that the question (A) must be answered by a reference to the consideration received for the risk.

In the case of £100,000 insured on the lives of 100 persons, if the premium for the risk is 1 per cent., or £1,000 in all, the premiums received will not generally be less than £1,250. But the net premium for an insurance of £100,000 on either of the very improbable contingencies mentioned in (A), would be an excessively small quantity; and the whole profit on the transaction so small, as to render it quite undesirable to entertain such a proposal

and incur so large a risk. If, however, circumstances permitted of the same profit—say £250—being made on the single £100,000 policy as on the 100 other policies, it might become a question whether it would not be worth while for the Company to undertake the risk.

We may perhaps state more broadly the principle here suggested, and say that when the profit on a transaction is unusually large, an Office may with propriety take the risk of a larger amount than in ordinary cases. Thus the premiums received upon "issue insurances" yield a far larger margin of profit than the premiums in ordinary life insurances; and a Company might perhaps, on that ground, properly take a larger risk in those cases than it does in life insurances.

As a rule, then, we may say that (1.) an Office will not run the risk of £100,000 on a single life, because the probability of loss is too great to be prudently incurred.

(2.) That there is no objection to the Office becoming liable for £100,000 upon 100 or 1,000 different lives, because the probability of a dangerous loss occurring is small, and the profit on the transaction is adequate to the risk.

(3.) That the Office would decline to grant a policy for £100,000 on an extremely improbable contingency, although the chance of loss is very small, because the profit on the transaction must generally be too small to be worthy of regard.

The previous principles enable us to answer a question which arises sometimes when a Company issues its policy for a large amount, but retains only a portion of the risk, reinsuring the remainder. In this case, can the Office prudently issue its acceptance for the whole amount before it has arranged to be relieved of the excess of risk? We must say, certainly not! For the consideration the Company receives for running the whole risk for a time, is so small as to be unworthy of regard.

Having now satisfactorily ascertained the reasons for which it is necessary to place a limit on the amount of the risks undertaken by an Office, we proceed to consider how the particular limit suitable for any Office is to be ascertained.

For this purpose we will suppose that a policy of the maximum amount has become a claim, and will consider out of what fund the claim is to be paid. For the present, we will, as before, suppose all the policies to be effected for a single year, in which case the claims of any year are to be paid out of the premiums received in the year. Let us suppose also that the rate of mortality is one per

cent. per annum, and examine how matters would stand if the business of the Office consisted of a few large policies—say of £10,000 each, and a large number of policies of £100 each. Then, if the premiums are loaded 25 per cent., the premium on each £10,000 policy will be £125; and on each £100 policy, £1·25; and if a policy for £10,000 becomes a claim, it will require the premiums of 80 such policies to satisfy the claim. If there are not so many £10,000 policies in force, the deficiency will have to be made good out of the premiums on the £100 policies; for example, if there are only 40 policies of £10,000, the premiums on these amount to £5,000, and it will require the premiums on 4,000 of the £100 policies to make up the other £5,000. But the premiums on the £100 policies are required to provide for the claims among those policies, and cannot therefore (except as regards the margin thereof) be considered available for payment of the £10,000 policy.

We thus arrive at the principle that the policies of maximum amount in an Office must be looked upon as forming a class by themselves; so that the claims arising among those policies are to be paid out of the premiums received upon them. It follows at once that the position of an Office cannot be considered satisfactory unless the policies of maximum amount are at least so numerous that the premiums received on them in the course of a year suffice to pay a claim, if one should arise; and the average premium being about 3 per cent., the very least number of maximum policies which an Office must have before its position can be declared satisfactory is 33. And unless the number of such policies is much greater than this, the Office will be liable to inconvenient fluctuation in the amounts of its annual claims. Supposing, first, that all the policies are effected for a single year only; then, if no claim occurs in the year, the whole of the premiums on the 33 policies will be profit; while if there should be a claim, it would absorb all the premiums, and there would be no profit on the year's transactions. This inconvenient fluctuation in the amount of the profits is to a very great extent remedied when a valuation for the distribution of profits is made only once in several years; for then it may be anticipated that the losses of one year will be compensated by the profits of others. Thus, if there are 33 policies of maximum amount under observation for three years before a valuation is made, this is, to a certain extent, the same as if there were 99 such policies under observation for a single year. On the whole, we may say that when an Office makes a valuation

and distribution of profits only once in three, five, or seven years; or when it makes an annual valuation, but distributes not the profit of the past year, but an amount obtained from the average of several years; then it may be considered as practically secure from inconvenient losses or fluctuation of profits, if it has about 33 policies of maximum amount in force.

But no Company can hope, when it adopts a particular maximum, to obtain at once so many policies of maximum amount. What, then, is the smallest number with which it should start? It is well known that the mortality among recently selected lives is very light, and that it is five or six years before the effect of selection wears off; we may then perhaps allow an Office that period for obtaining the above mentioned number (33) of maximum risks. Thus we finally conclude that an Office ought yearly to obtain *at least* six or seven policies of maximum amount; and if the Office does not obtain that number, it must be inferred that its maximum has been fixed too high and should be lowered.

When the business of an Office is very extensive, the previous conclusions will require to be modified. Resuming the supposition of a mortality of 1 per cent., and supposing that the premium on a £100 policy is 1·25, we may say in general terms that the £1 is required to pay the claims, and the ·25 is available for expenses and contingencies; or we may consider that, after payment of expenses, the balance of the loading (·25) is available for payment of the £10,000 claims. If, then, the business is so extensive that the margins of the premiums, after payment of expenses, are equal to say three or four times the maximum amount of risk, we see that a claim of maximum amount might be paid out of the margins of the premiums on the smaller policies, without any greater inconvenience than a temporary diminution of the profits of the Office; and we thus arrive at the conclusion, that when the business of an Office is so extensive that the annual anticipated profit, will pay without inconvenience a maximum claim, then it is not essential that the maximum policies should be numerous. A Company of *very* large business might even prudently issue a single policy much larger than any other of its existing policies.

We have supposed above that all the policies were effected for a single year; but our conclusions are equally applicable to the ordinary insurances of a Life Office, if we substitute for the annual margins of the premiums, the annual anticipated profit, which may be stated in a general way to consist of the margins of the premiums, and of the excess of the interest realized over the rate of

interest—3, 3½, or 4 per cent.—assumed in the calculations of the Office. But it must be borne in mind that if, in making a distribution of profits, the margin of the premiums has to any extent been encroached upon, it is to that extent non-existent.

The conclusion at which we have arrived above, viz., that the largest policies in an Office are generally to be considered as a class by themselves—so to speak, self-supporting—appears to sum up all that can be said on our present subject. Notwithstanding its importance, it is often overlooked by actuaries. Thus we sometimes hear the remark made, that "the claims of the year have fallen on large amounts." In this form of expression it is implied that a certain number of claims, large or small, was to be expected in the year; and that large claims have occurred where small ones were to have been expected; but there is clearly no reason for supposing that, because the claims among the small policies have been less numerous than might have been expected, therefore the claims among the large policies should be more numerous than the expectation. There is, in truth, no possible relation between the number of claims in the two classes of policies—large and small; and under the circumstances supposed, it would be more correct to say simply, that an unexpected number of deaths has occurred among the large policies.

So, again, I have heard it suggested that an Office might fix its maximum risk at such an amount, that if all the anticipated claims of the year were of the maximum amount, the total claims would just absorb a year's income. But this rule implies, that the more numerous the small policies are in an Office, the greater is the risk of each of the large policies becoming a claim, which is manifestly absurd. We conclude, on the contrary, that the amount of the maximum risk is to be fixed independently of the income or assets of the Office, by reference to the number of policies of maximum amount which it is probable will be effected—subject to the modification stated previously, when the income of the Office is *very* large.

I have spoken hitherto of the division of insurances simply into large and small; but, in practice, there will always be insurances of intermediate amount, more or less numerous; and in applying our rule, it will not be necessary to confine our attention to the insurances of the precise maximum amount, but we may fairly class with them such as do not much fall short of that amount. Thus, if the premiums are loaded 25 per cent., so that the Office premium for £4,000 is equal to the net premium for £5,000, we may fairly

group together all the insurances for £4,000, £5,000, and intermediate amounts; and so with the insurances from £8,000 to £10,000; so that if the maximum of an Office is £5,000, the position of the Office will be considered satisfactory, if the number of insurances ranging from £4,000 to £5,000 is sufficient to give an average.

With regard to the particular limit to be fixed by an Office to its risks, it may be suggested that if the Office wishes to keep more than £5,000 it is scarcely worth while to stop short of £10,000, as the insurances of intermediate amounts are not very numerous. There is much weight in this remark, when it is the practice of the Office to retain the whole of the risk under all the policies it grants; but the practice of "reassurance," by which Offices interchange risks the one with the other, enables an Office to fix on any particular limit that may for the time being seem advisable. In the early years of an Office, it will seldom be advisable to fix a higher limit than £2,000 or £3,000; and as the number of large cases increases, the limit can be increased by steps of £1,000 to £5,000, £6,000 . . . £10,000, and even further, if justified by circumstances. The Office, while retaining only £2,000 at its own risk, may legitimately issue its policy for £5,000, and "reinsure" the balance, £3,000, with other Offices, which may be expected to make it a return. In this way, while the large risks of the Office are few, its position is greatly improved, since the probability of the occurrence of an inconvenient loss is greatly diminished.

As far as I can learn, no Insurance Company at the present time retains more than £15,000 at risk upon a single life; and I am only aware of one Office which retains so large an amount. But, if the arguments given above are valid, there would seem to be no objection to an Office keeping £20,000, £25,000, or £30,000 on one life, provided it can get a sufficient number of similar insurances, *i.e.*, not less than at least six or seven each year.

I now pass on to the consideration of various questions more or less connected with our general subject.

I have first to remark, that when a policy is effected for the whole term of life, the policy usually has a "value," which increases from year to year; and when we examine the effect this fact produces on the financial course of an Office, we see that in the first year of an insurance any claim arising must be paid out of the premiums received in the year; but in every subsequent year there is a reserve appertaining to each policy, which is available towards

payment of the claim when it arises. We may say, then, that when a claim arises in a year subsequent to the first, a part of the sum assured—equal to the value of the policy—is paid out of the accumulated funds; and the balance only is paid out of the income —premiums and interest. The loss of the Office is confined to the latter amount; the payment as regards the value of the policy being more in the nature of the repayment of a loan. Thus, then, the risk of the Office decreases year by year, being only equal to the excess of the sum assured over the value of the policy. It follows that when a life has been for some time insured for the maximum amount in an Office, a further insurance may be accepted for a sum equal to the value of the existing policy, without increasing the risk of the Office beyond the usual maximum.

In illustration of this principle, let us suppose that an Office whose usual maximum is £5,000, has, through an oversight, been for some years exposed to a risk of double that amount on a particular life. Thus, a policy for £5,000 may have been granted on the life of James Duff, M.P., and another policy for £5,000 may subsequently be granted on the life of the Earl of Fife; and several years may elapse before it is observed that these are different names of the same person. When at last the discovery is made, what sum should be reinsured in order to reduce the risk of the Office to its usual maximum? If V_1 and V_2 are the values of the two policies, we see, by applying the principles laid down above, that the actual risk of the Office is £10,000$-V_1-V_2$; so that the sum to be reinsured will be £5,000$-V_1-V_2$.

Next, what is the risk of an Office under a *reversionary annuity*; as, for example, when an annuity of £1,000 is assured to be paid to x for life after the death of y? Here, if y die during the first year, the Office becomes liable for an annuity of value $1{,}000 \times a_x$; and this therefore must be considered as the amount of the risk in the first year. The amount of the risk in the second year is similarly $1{,}000 \times a_{x+1}$, which is less than the risk in the first; and the risk becomes less each year that the policy is in force. If such a policy is effected by means of a uniform annual premium, it will be found that the policy has no value, or a negative one, at the end of any number of years from the date of the assurance.

It may be suggested that in this case the usual maximum may be properly exceeded, because the Office has not, in the event of a claim, to pay a large sum down, but only to make the various payments of the annuity as they fall due. But this reasoning appears to me inadmissible; for we must consider what is the

amount of the loss which would have to be provided for in a valuation; and the amount of the sum payable at the time of the claim is a matter of very secondary importance. I therefore conclude that any Office would act unwisely if it retained a larger risk under a reversionary annuity than it would do under an ordinary policy.

These examples may suffice to show that the risk of an Office, or the amount which it is liable to lose in the event of death, is often a very different thing from the payment it will be called upon to make. But this distinction is often lost sight of. Thus, it is the practice of some Offices to retain a larger sum at risk under a policy that is effected without profits, than is retained under a participating policy; on the ground that the bonuses may considerably increase the sum payable at the time when the claim arises. But if, as before, we consider the "risk" of the Office, which is only the excess of the sum assured and the bonus over the value of the same, we see that in practice it can seldom happen, however large the bonus may be, that the risk of the Office shall exceed the original sum assured; and we therefore conclude that there is no sufficient reason for retaining a less sum at risk on a bonus policy than on a non-participating policy.

From the same considerations it follows that when an insurance is effected on the last of two or three lives, the Office may properly retain at its own risk a larger amount than it would retain on a single life. At first sight we might suppose that a double sum assured could properly be retained on the last of two lives. "For," it may be said, "in the case of an insurance for double the usual maximum on the last of two lives, the risk is divided between the two, and the risk on either life is only equal to that usually undertaken by the Office. Thus, when one of the lives dies, although no cash payment has to be made by the Office, yet a loss has occurred, which has to be provided for in a valuation; and when the second life dies, although a double sum becomes payable, yet one half has been already provided for; and the loss, properly so called, consequent on the death, is no larger than in the case of an insurance of the maximum amount on a single life." But when we examine more closely, we shall find that this reasoning is only correct on the supposition that the premium for an insurance on the last of two lives is one half that for an insurance on one of the lives. For, in general, the risk is not *equally* divided between the two lives; but when one of the lives dies, the consequent loss is less than half the sum assured. For example,

suppose that an insurance for £10,000 is granted on the last survivor of two lives, each 40 years of age, at a premium of £2 per £100—the premium for an insurance on a single life of 40 being £3 per £100; then, if one of the lives dies in the first year of the insurance, what is the consequent loss sustained by the Office? The annual premium, £200, would insure a sum of £6,666. 13*s.* 4*d.* on the remaining life; but the sum now payable on the dropping of that life is £10,000, so that we may say that £6,666. 13*s.* 4*d.* is provided for by the premium receivable, but the remaining £3,333. 6*s.* 8*d.* is unprovided for; and is, in fact, the loss which has occurred in consequence of the failure of the first life. In this case, then, the Office would lose £3,333. 6*s.* 8*d.* by the death of the first life, instead of £5,000 as assumed in the foregoing reasoning; and would then have £6,666. 13*s.* 4*d.* at risk on the second life, its usual maximum being £5,000. We shall determine the amount which may be properly insured on the last of the two lives as follows:—The annual premium on an insurance of £5,000 on one of the lives is £150, which will insure £7,500 on the last of the two lives; and that therefore is the sum which can be insured on the last of two lives of the age of 40, without exposing the Office to a larger risk than £5,000 on a single life.

Next, suppose that there are two lives of 25 and 50; that the premiums at those ages are £2 per £100, and £4 per £100; and that the premium for an insurance on the last of 25 and 50 is £1. 15*s.* per £100. As before, suppose that £10,000 is insured on the last of 25 and 50, at an annual premium of £175. This premium will insure £8,750 on a life of 25, and £4,375 on a life of 50. Hence we see that if 50 dies in the first year of the insurance, the risk left depending on 25 is £8,750, and the loss is only £1,250; while if 25 die, the sum depending on 50 is £4,375, and the loss is £5,625. To get the sum that may be insured properly on the last of 25 and 50, we take the premium for £5,000 on a life of 25 (*i.e.*, £100), and since this will insure £5,714 on the last survivor of 25 and 50, that is the largest sum that ought to be insured on the last of the two lives. If x and y are the two lives, of which x is the younger, π_x the annual premium for an insurance of £1 on x, $\pi_{\overline{x.y}}$ the premium for the same insurance on the last of x and y; then the amount which may be insured on the last of the two lives is $\frac{\pi_x}{\pi_{\overline{x.y}}} \times \mathrm{S}$, where S is the maximum sum insured on a single life.

If, however, the lives are closely related, the risks cannot be

considered as wholly independent; and it would scarcely be prudent to incur the risk of the larger sum on the last survivor of two closely related persons.

So again, we see that an Office might grant an "endowment" to be payable only in case a nominated person attain an assigned age, of a much larger amount than any insurance it would undertake; for the possible loss, in consequence of the life attaining the assigned age, will be only a fraction of the sum assured.

Another practical question arising in the course of life insurance business is this—Shall an Office take the same amount of risk in an old life as it takes in a young one? This question is to be answered in conformity with the principles we have laid down, by reference to the comparative number of policies on the lives of old and young persons. But it must be borne in mind, that we need not restrict the comparison to the insurances effected on young and old lives; for we may group together all the policies existing on old lives, whatever their duration. If, then, we should find that the insurances on the lives of old persons are not sufficiently numerous to form an average among themselves, it may, perhaps, be prudent to insure a smaller sum on the life of an old person than on a young one. It must be considered, on the other hand, that the test of medical examination is much more efficacious at the older ages than at the younger. It must also be remembered that since the premium on an old life is higher than on a young one, the number of old lives sufficient to give an average is less than the number of young lives required for the same purpose. In actual practice, we may probably say that there is no sufficient reason for insuring a less sum on the life of an old man than on the life of a young man.

This question has been considered by Mr. Scratchley, who has given a different answer to it. He says, in his *Treatise on Life Assurance Societies* (p. 36, App.)—"If £3,000 be the amount of policy granted on a life aged 30, not more than £1,677 should be granted on a life aged 60, otherwise the Society would experience *moral* disadvantage; for it is obvious that although from the premiums being larger on policies in advanced ages, the value of the *mathematical* expectation of the ultimate payments received by a Company from an assurance on an old life may be equal to that from a corresponding assurance on a younger and consequently longer enduring life, yet the *moral* expectations of the immediate risks for any current period can only be rendered equal to each other, or nearly so, by a large average of about the same age, or

by some system of graduation of the amount assured, so as to tally with the increase of immediate risk attending the fact of the ages being older." Accordingly, he gives a table of the amounts to be insured at various ages. The above reasoning, however, appears to be erroneous, inasmuch as it is independent of the *relative* numbers insured at the various ages. If the number of persons insuring at the age of 60 were equal to those insuring at 30, I have no hesitation in saying, that the Office might safely retain a *larger* sum at risk on the old lives than on the young.

With regard to the so-called "moral" expectation, mentioned by Mr. Scratchley above, I have to confess that I am unable to see that the theories on this subject are of the least practical utility. I have preferred, throughout this paper, to treat the various questions considered independently of those theories; and I trust that my conclusions will be found none the less satisfactory on this account.

I have now only to remark, in conclusion, that I am well aware that my subject is far from exhausted by the preceding remarks, and that much remains to be done to complete the mathematical part of the inquiry. I trust, however, that this attempt to elucidate some questions hitherto very little discussed, will be found of some practical utility to those engaged in the management of Life Insurance Societies.

General Average. *By* Richard Morrison.

(Continued from page 361, vol. xii.)

In considering the various expenses, or, as some writers term them, expenditures, that are the results of a general average act, the same test must be applied as in the case of sacrifices. Were they had recourse to as being, in the exercise of sound judgment and prudence, indispensably necessary for procuring the safety of all concerned in the adventure? Might the act from which they proceeded be the means of arresting any course of events which, if allowed to take their full scope, would terminate in the destruction of the vessel and the consequent loss of the freight and cargo? It is true human judgment is likely to err under these, as in other circumstances of like nature, and expenses may be incurred needlessly, for the ultimate consequences of an act cannot be always perceived; but if the end proposed will admit of no other course by which the vessel and the property in her may be preserved at